THE UNIVERSITY OF ALBERTA

APPROXIMATION BY RATIONAL INTERPOLATION OPERATORS

BY

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A THESIS

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ABSTRACT

In this thesis, rational interpolation operators are constructed from polynomial interpolation operators in such a way that the rate of convergence is improved. A short description of the method to be used is contained in Chapter 1.

Chapters 2 and 3 deal with the background material required to apply the method. A Korovkin-type theorem which appears in Chapter 2 forms the means to determine rates of convergence for the rational interpolation processes. The required information regarding the polynomial interpolation processes to be used is in Chapter 3.

The new interpolation processes are constructed in Chapters 4 and 5, and their rates of convergence determined. The results in Chapter 4 are of a more general nature, while those of Chapter 5 yield a better degree of approximation.

Chapter 6 contains some suggestions for ways that these results could be extended or improved.

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CHAPTER 1

INTRODUCTION

The problem of finding interpolation processes which provide good tools for approximation is one which has been well studied, and a number of such processes have been found. It is natural to attempt to modify known processes in order to improve on the degree of approximation obtained. This thesis contains a number of such attempts. The general method of modification is natural enough, and could likely be applied to interpolation processes other than those considered in this thesis:

Let H_n be a positive linear interpolation operator defined on $C[-1,1]$ by

$$H_n f(x) = \sum_{k=1}^{n} g_{nk}(x) f(x_{nk}) ,$$

where $-1 \le x_{nn} < \cdots < x_{n1} \le 1$ and the g_{nk} are nonnegative continuous functions defined on $[-1,1]$. Suppose that

$$\sum_{k=1}^{n} g_{nk}(x) \equiv 1 ,$$

so that $H_n 1 = 1$. Then the degree to which H_n approximates continuous functions is largely dependent upon how rapidly $|g_{nk}(x)|$ decreases and $|x-x_{nk}|$ increases. This dependence is expressed precisely by Corollary 2.1.

Presented with such an operator H_n, where $|g_{nk}(x)| \to 0$ as $|x-x_{nk}|$ grows, one can often realize an improvement by considering

instead an operator L_n:

$$L_n f(x) = \frac{\sum_{k=1}^{n} m_{nk}(x) f(x_{nk})}{\sum_{k=1}^{n} m_{nk}(x)},$$

where $m_{nk}(x) = g_{nk}^2(x)$, or something very like $g_{nk}^2(x)$. Then $m_{nk}(x)$ will approach zero more rapidly than $g_{nk}(x)$ as x moves away from x_{nk}. Thus, L_n will be superior to H_n as a tool of approximation, provided that the minimum value achieved by the denominator does not approach zero too quickly (or, in the best case, is bounded away from zero). L_n does represent an increase in complexity over H_n - e.g., if H_n is a polynomial operator then the images of L_n are rational functions. However, the denominator of $L_n f$ is independent of f, and so need be calculated only once for each n.

This method will be applied to operators connected with Hermite-Fejér and related interpolation processes, which will be discussed in Chapter 3. A basic paper in this field of study is Fejér [5]. The precise effectiveness of the H-F process was discovered by R. Bojanic, and is given by DeVore [4, p.232]. Questions concerning slight modifications of the process were considered by Turán [11]. Other modifications and extensions were defined and studied in Balázs and Turán [1,2,3]; Meir, Sharma, and Tzimbalario [6]; Prasad and Varma [7], and Sharma and Tzimbalario [8].

The nodes of the processes considered will be the zeros of the classical ultraspherical polynomials. Information about these can be found in Szegö [10], and will be given where it is required.

CHAPTER 2

A KOROVKIN-TYPE THEOREM

In this chapter will be stated and proven a Korovkin-type theorem due to Mond and Shisha, and then the result will be restated in a form in which it can be applied to the operators to be considered.

Definition 2.1: Let L be a linear operator defined on a set D of real-valued functions s.t. for every $f \in D$, Lf is again a real-valued function, and if $f \geq 0$ throughout its domain, then $Lf \geq 0$. Then L is a positive linear operator.

Remark: Both notations $Lf(x)$ and $L(f;x)$ will be used to denote the function Lf evaluated at x.

Definition 2.2: If $f \in C[a,b]$, the modulus of continuity ω_f of f on $[a,b]$ is defined by

$$\omega_f(\delta) = \sup\{|f(t)-f(x)| \ |x,t \in [a,b], |x-t| \leq \delta\} \quad \text{for} \quad \delta > 0$$

Remark: ω_f is nondecreasing on $(0,\infty)$, $\lim\limits_{\delta \to 0+} \omega_f(\delta) = 0$, and $\omega_f(\gamma\delta) \leq (\gamma+1)\omega_f(\delta)$ for all $\gamma,\delta > 0$.

The operators to be considered will be of a certain type which R. DeVore [2] calls interpolation operators.

Definition 2.3: Let L be a linear operator from $C[a,b]$ to $C[a',b']$ s.t. there exist $x_1,\cdots,x_n \in [a,b]$ and $g_1,\cdots,g_n \in C[a',b']$ so that for every $f \in C[a,b]$ and $t \in [a',b']$,

$$Lf(t) = \sum_{k=1}^{n} g_k(t)f(x_k).$$

Then L is called an interpolation operator, or a proper interpolation operator if $[a',b'] \subset [a,b]$. Also, $x_1,\cdots,x_n$ will be called the nodes of L.

Essentially, then, an interpolation operator is one for which the image of a function is determined by the values of the function on a certain fixed finite set of points. Examples of interpolation operators are provided by the processes of Lagrange, Hermite-Fejer, extended Hermite-Fejer, and other forms of interpolation. It should be noted that, if L is a proper interpolation operator, it will not necessarily interpolate the values of every function at the points $x_1,\cdots,x_n$ (this occurs only when $g_k(x_j) = \delta_{kj}$ for all k,j); however, most of the operators considered here will do so.

The following theorem, due to B. Mond and O. Shisha, will be used in determining the rates of convergence for certain interpolation operators in later chapters and represents, in a sense, the center of this thesis.

Theorem 2.1 (O. Shisha and B. Mond [9]): Let L be a positive linear operator from $C[a,b]$ to $C[c,d]$ whose domain includes the functions 1, t, and t^2, with $L1 = 1$. If $[c,d] \subset [a,b]$ and $\|\cdot\|_\infty$ denotes the sup norm on $[c,d]$, then, for every $f \in D$,

$\|Lf-f\|_\infty \le 2\omega_f(\mu)$, where $\mu = \|L((x-t)^2;x)\|_\infty^{1/2}$ and ω_f is the modulus of continuity of f on $[a,b]$.

Remark: In forming $L((x-t)^2;x)$, L is applied to $(x-t)^2$ as a function of t, and the image is evaluated at x.

Proof of Theorem 2.1. Let $x \in [c,d]$, $t \in [a,b]$, and $\delta > 0$. Then $|f(t)-f(x)| \le (1 + (x-t)^2\delta^{-2})\omega_f(\delta)$ (trivial if $|x-t| \le \delta$, and if $|x-t| > \delta$ then $|f(t)-f(x)| \le \omega_f(|x-t|\delta^{-1}\delta) \le (1 + |x-t|\delta^{-1})\omega_f(\delta) \le (1 + |x-t|^2\delta^{-2})\omega_f(\delta)$.) Applying L to both sides of this inequality (holding x fixed) and evaluating the results at x yields:

$$|Lf(x)-f(x)| \le (1 + L((x-t)^2;x)\delta^{-2})\omega_f(\delta).$$

The theorem then is proven by putting $\delta = \mu$.

A restatement of this theorem, in the context of interpolation operators, follows:

Corollary 2.1: Let $x_1,\cdots,x_n \in [a,b]$, and let $m_1,\cdots,m_n$ be nonnegative continuous functions defined on a subinterval $[c,d]$ of $[a,b]$.

Let $p = \sup\limits_{x\in[c,d]} \dfrac{\sum_{k=1}^{n} m_k(x)(x-x_k)^2}{\sum_{k=1}^{n} m_k(x)}$, and define, for $f \in C[a,b]$,

$$Lf(x) = \frac{\sum_{k=1}^{n} m_k(x)f(x_k)}{\sum_{k=1}^{n} m_k(x)}.$$

L is then a positive linear operator and if $f \in C[a,b]$ with modulus of continuity ω_f and $\|\cdot\|$ denotes the sup norm on $[c,d]$, then

$$\|Lf - f\|_\infty \leq 2\omega_f(\sqrt{p}).$$

Remark: Notice that L, defined in this way, is a proper interpolation operator - that is, $Lf(x) = \sum_{k=1}^{n} g_k(x)f(x_k)$, with $g_k(x) = \dfrac{m_k(x)}{\sum_{j=1}^{n} m_j(x)}$.

Moreover, $Lf(x_k) = f(x_k)$ whenever $m_k(x_j) = \delta_{kj}$ for all k,j.

A simple extension of this corollary will be used to extend a result in Chapter IV.

Theorem 2.2: Let $x_1,\cdots,x_n \in [a,b]$ and let $m_1,\cdots,m_n$ be nonnegative continuous functions defined on $[c,d] \subset [a,b]$. Let $\|\cdot\|_\infty$ denote the sup norm on $[c,d]$ and let $\gamma \geq 1$. Let $L:C[a,b] \to C[c,d]$ be defined by

$$Lf(x) = \frac{\sum_{k=1}^{n} m_k(x)f(x_k)}{\sum_{k=1}^{n} m_k(x)}.$$

Then

$$\|Lf-f\|_\infty \leq 2\omega_f\left(\|L(|x-t|^\gamma;x)\|_\infty^{1/\gamma}\right).$$

The proof of this theorem is essentially similar to that of Theorem 2.1.

CHAPTER 3

HERMITE-FEJER AND RELATED INTERPOLATION PROCESSES

In this chapter various interpolation processes and their basic polynomials will be introduced. This information will be used in Chapter 5 to construct certain interpolation operators which will be shown to yield a good degree of approximation.

Definition 3.1. A polynomial interpolation process on an interval $[a,b]$ is defined to be a sequence $\{L_n\}$ of interpolation operators such that for every $n = 1,2,\cdots$ there exist polynomials $g_{n1},\cdots,g_{nn}$ and nodes $x_{nn} < \cdots < x_{n1}$ so that for $f \in C[a,b]$,

$$L_n f(x) = \sum_{k=1}^{n} g_{nk}(x) f(x_{nk}), \quad x \in [a,b].$$

The g_{nk} are called the basic polynomials of the process.

A number of examples of polynomial interpolation process are listed below, and the basic polynomials are given in each case.

Lagrange Interpolation: Lagrange interpolation is perhaps the simplest form of polynomial interpolation by which, for an integer $n \geq 1$ and a continuous function $f(x)$, a polynomial $p \in \pi_{n-1}$ is determined s.t. $p(x_{nk}) = f(x_{nk})$ where $x_{n1},\cdots,x_{nn}$ are the distinct fixed nodes. That is, one defines the Lagrange polynomial interpolation process $\{L_n\}$ with nodes $x_{n1},\cdots,x_{nn}$ in $[-1,1]$ so that if $f \in C[-1,1]$, then $L_n f(x_{nk}) = f(x_{nk})$, $k = 1,\cdots,n$ and $L_n f \in \pi_{n-1}$. The basic polynomials of this process are

$$\ell_{nk}(x) = \frac{\omega_n(x)}{(x-x_{nk})\omega_n'(x_{nk})}, \quad \text{where} \quad \omega_n(x) = \prod_{k=1}^{n} (x-x_{nk}),$$

since then $\ell_{nk}(x_{nj}) = \delta_{kj}$ and $\ell_{nk} \in \pi_{n-1}$.

Hermite-Fejer (H-F) Interpolation: The H-F process $\{L_n\}$ with nodes $\{x_{nk}\}$, $-1 \le x_{nn} < \cdots < x_{n1} \le 1$ $(n = 1,2,\cdots)$, is defined by:

$$L_n f(x) = \sum_{k=1}^{n} h_{nk}(x) f(x_{nk}) \quad \text{for} \quad f \in C[-1,1], \quad x \in [-1,1],$$

where $h_{nk}(x) = \left(1 - (x-x_{nk}) \frac{\omega_n''(x_{nk})}{\omega_n'(x_{nk})}\right) \ell_{nk}^2(x)$, $\ell_{nk}(x) = \dfrac{\omega_n(x)}{(x-x_{nk})\omega_n'(x_{nk})}$,

and $\omega_n(x) = \prod_{k=1}^{n} (x-x_{nk})$, so that $L_n f \in \pi_{2n-1}$.

Simple calculations reveal that, for all k,j,

$$h_{nk}(x_{nj}) = \delta_{kj} \Rightarrow L_n f(x_{nk}) = f(x_{nk})$$

$$h'_{nk}(x_{nj}) = 0 \Rightarrow (L_n f)'(x_{nk}) = 0.$$

Extended H-F Interpolation: This process extends the previous one by requiring also that the second and third derivatives of $L_n f$ vanish at the nodes x_{nk}, and is defined by:

$$L_n f(x) = \sum_{k=1}^{n} \lambda_{nk}(x) f(x_{nk}),$$

where

$$\lambda_{nk}(x) = \left(1 + C_1(x-x_{nk}) + C_2(x-x_{nk})^2 + C_3(x-x_{nk})^3\right) \ell_{nk}^4(x),$$

$$C_1 = -4\ell'_{nk}(x_{nk})$$

$$C_2 = 10\left(\ell'_{nk}(x_{nk})\right)^2 - 2\ell''_{nk}(x_{nk})$$

$$C_3 = -20\left(\ell'_{nk}(x_{nk})\right)^3 + 10\ell'_{nk}(x_{nk})\ell''_{nk}(x_{nk}) - \frac{2}{3}\ell'''_{nk}(x_{nk}).$$

Once again calculations yield that

$$L_n f \in \pi_{4n-1},$$

$$L_n f(x_{nk}) = f(x_{nk}) \quad \text{and} \quad (L_n f)'(x_{nk}) = (L_n f)''(x_{nk}) = (L_n f)'''(x_{nk}) = 0.$$

Remarks: 1) $\sum_{k=1}^{n} \ell_{nk}(x) = \sum_{k=1}^{n} h_{nk}(x) = \sum_{k=1}^{n} \lambda_{nk}(x) = 1$ is a trivial consequence, in each case, of the properties imposed on the basic polynomials and their degrees as polynomials.

2) It will be the task of Chapter 5 to use information that has been discovered relating to certain special cases of polynomial interpolation processes in order to derive "good" methods of linear rational approximation. There follows a description (in some generality) of the underlying methods to be used in each case.

The problem is to find certain polynomials $g_{nk}(x)$, and nodes x_{nk} $(-1 \leq x_{nn} < \cdots < x_{n1} \leq 1)$, $n = 1,2,\cdots$, $1 \leq k \leq n$, s.t., for some interval $I \subset [-1,1]$,

(i) $g_{nk}(x) \geq 0$ for all n,k and $x \in I$

(ii) there exists $c > 0$ so that

$$c \leq \sum_{k=1}^{n} g_{nk}(x) \quad \text{for all } n \text{ and } x \in I$$

and (iii) $\sup_{x \in I} \left| \sum_{k=1}^{n} g_{nk}(x)(x-x_{nk})^2 \right| = o(1)$, as $n \to \infty$.

A sequence $\{L_n\}$ of linear operators on $C[-1,1]$ whose images are rational functions will be defined by

$$L_n f(x) = \frac{\sum_{k=1}^{n} g_{nk}(x) f(x_{nk})}{\sum_{k=1}^{n} g_{nk}(x)} .$$

The Korovkin method (Corollary 2.1) will be then applied to these operators to get an estimate for $\|L_n f - f\|_\infty$.

CHAPTER 4

A GENERAL RATIONAL INTERPOLATION APPROXIMATION RESULT

The main results of this chapter establish certain properties of the functions $h_{nk}(x)$ (not necessarily polynomials), $n = 1,2,\cdots$, $k = 1,\cdots,n$, which ensure that certain approximation procedures involving these h_{nk} yield reasonably good degrees of approximation.

<u>Definition 4.1</u>: Let $[a,b]$ be a fixed interval and suppose, for $n = 1,2,\cdots$, that points x_{nk} $(k = 1,\cdots,n)$ are given and there exist $\eta > 0$ and integers n_o, N s.t. for every interval $I \subset [a,b]$ of length $\ell(I)$ and for every $n \geq n_o$,

$$\text{card}\,\{k | x_{nk} \in I\} \leq \eta \cdot \ell(I) \cdot n + N;$$

then the points $\{x_{nk}\}$ are said to be "η-dispersed in $[a,b]$".

<u>Theorem 4.1</u>: Let points $-1 \leq x_{nn} < \cdots < x_{n1} \leq 1$ and functions $h_{n1}(x),\cdots,h_{nn}(x)$ be given $(n = 1,2,\cdots)$, and suppose there exist constants $\eta > 0$, $M > 0$, $\varepsilon > \mu > 0$, $C_2 \geq C_1 > 0$, and an integer n_1 s.t. for all $x \in I_\varepsilon = [-1+\varepsilon,1-\varepsilon]$, the following conditions hold:

(1) $h_{nk}(x) \geq 0$, all $n \geq n_1$, all $k = 1,\cdots,n$

(2) $C_1 \leq \sum_{k=1}^{n} h_{nk}(x) \leq C_2$, all $n \geq n_1$

(3) $h_{nk}(x) \leq \dfrac{M}{n^2(x-x_{nk})^2}$, all $n \geq n_1$, all $k = 1,\cdots,n$

(4) The points $\{x_{nk}\}$ are η-dispersed in $I_\mu = [-1+\mu, 1-\mu]$.

Define a sequence L_n of positive linear operators on $C[-1,1]$ by:

$$L_n f(x) = \frac{\sum_{k=1}^{n} h_{nk}^2(x) f(x_{nk})}{\sum_{k=1}^{n} h_{nk}^2(x)}, \quad x \in I_\varepsilon$$

Then, if $\|\cdot\|_\infty$ denotes the sup norm on I_ε,

$$\|L_n f - f\|_\infty \le C\omega_f\left(\frac{\sqrt{\ell n\ \ell n\ n}}{n}\right),$$

where C is a constant independent of n and f, and ω_f denotes the modulus of continuity of f on $[-1,1]$.

Remarks: This result is, in a certain sense, a generalization of Theorem 5.1. The degree of approximation obtained in the theorems of Chapter 5 is $0\left(\omega_f(\frac{1}{n})\right)$, while the result in Theorem 4.1 is the degree $0\left[\omega_f\left(\frac{\sqrt{\ell n\ \ell n\ n}}{n}\right)\right]$. The factor $\sqrt{\ell n\ \ell n\ n}$ converges to ∞ as $n \to \infty$; however, numerically, it grows rather slowly. For example, if $n = 10^{10^{10}}$, still $\sqrt{\ell n\ \ell n\ n} < 5$.

The proof of Theorem 4.1 requires some lemmas:

Lemma 4.1: Let points x_{nk}, functions h_{nk} $(n = 1,2,\cdots;\ k = 1,\cdots,n)$, constants $\eta > 0$, $M > 0$, $\varepsilon > \mu > 0$, and $C_2 > C_1 > 0$, and an integer n_1 be given, so that assumptions (1) to (4) of Theorem 4.1 are satisfied. Then

(i) if $\alpha > 0$ there exists an n_2 and $A_\alpha > 0$ s.t.

$$n \ge n_2,\ x \in I_\varepsilon \Rightarrow n^\alpha \sum_{k=1}^{n} h_{nk}^2(x) \ge A_\alpha$$

and (ii) if $\{\alpha_n\}$ is a sequence of positive reals s.t. $\alpha_n \downarrow 0$ and $n^{-(1/2)}|\ell n\alpha_n| \to 0$, then for some n_3 and $A > 0$,

$$n \geq n_3,\ x \in I_\varepsilon \Rightarrow n^{\alpha_n}|\ell n\alpha_n| \sum_{k=1}^{n} h_{nk}^2(x) \geq A.$$

Lemma 4.2: Under the assumptions of Lemma 4.1, there exists a constant $B > 0$ and an integer n_4 s.t. whenever $n \geq n_4$, $x \in I_\varepsilon$, we have

$$\frac{\sum_{k=1}^{n} h_{nk}^2(x)(x-x_{nk})^2}{\sum_{k=1}^{n} h_{nk}^2(x)} \leq \frac{B\ \ell n\ \ell n\ n}{n^2}.$$

Remarks: Part (i) of Lemma 4.1 is actually implied by part (ii), but the statement and proof of part (i) are given since they form the basis for part (ii), and the proof is simpler in the former case.

It should be noted that essentially the best choice for the α_n is $\alpha_n = (\ell n\ n)^{-1}$ - i.e., this choice gives essentially the slowest growth in the factor $n^{\alpha_n}|\ell n\alpha_n|$. Suppose $\{\beta_n\}$ is any other choice s.t. $\beta_n > 0$, $\beta_n \to 0$, and $n^{\beta_n}|\ell n\beta_n| \leq Kn^{\alpha_n}|\ell n\alpha_n|$ for some $K > 0$; then $|\ell n\beta_n| \to \infty \Rightarrow n^{\beta_n} = O(\ell n\ \ell n\ n)$ (since $n^{\alpha_n}|\ell n\alpha_n| = \ell n\ \ell n\ n) \Rightarrow$ $\beta_n = O\left(\frac{\ell n\ \ell n\ \ell n\ n}{\ell n\ n}\right) \Rightarrow |\ell n\beta_n| \geq \frac{1}{2}\ \ell n\ \ell n\ n$ for n large enough, and hence $n^{\beta_n}|\ell n\beta_n| \geq \frac{1}{2}\ \ell n\ \ell n\ n$ for n large enough. Therefore (except for at most a constant factor), $\alpha_n = (\ell n\ n)^{-1}$ yields the best result.

From the above it is also clear that the requirement $n^{-(1/2)}|\ell n\alpha_n| \to 0$ does not weaken the result of Lemma 4.1, since certainly $n^{-(1/2)}|\ell n\alpha_n| \to 0$ for the choice $\alpha_n = (\ell n\ n)^{-1}$.

Proof of Lemma 4.1: It can be assumed w.l.g. that $C_1 \geq 1$. (If not,

divide by C_1.). Let $n \in Z^+$ and define $\nu_i = n^{-1+2^{-i}}$, $i = 1,2,\cdots$ so that $n\nu_i^2 = \nu_{i-1}$ and $\nu_{i-1} < \nu_i$ for all i. Let n_o, $N \in Z^+$ be as in Definition 4.1 (on the interval $[a,b] = [-1+\mu,1-\mu]$.).

<u>Part (i)</u>: Let $\alpha > 0$, and assume w.l.g. that $\alpha < 1$. Let $j \in Z^+$ s.t. $2^{-j} < \alpha < 2^{-j+1}$. Let $x \in I_\varepsilon$ and $C = \max\{1, 2j(2\eta+N)M\}$. Let $n_2 \geq n_o$ s.t. $n \geq n_2 \Rightarrow Cn^{-(1/2)} < \varepsilon - \mu$. Assume $n \geq n_2$. Define

$$I_1(x) = \{k \,|\, |x-x_{nk}| > C\nu_1\}$$

$$I_m(x) = \{k \,|\, C\nu_m < |x-x_{nk}| \leq C\nu_{m-1}\},\ m = 2,\cdots,j$$

$$I_{j+1}(x) = \{k \,|\, |x-x_{nk}| \leq C\nu_j\}$$

Then, for $m = 2,\cdots,j+1$,

$$\text{card}\big(I_m(x)\big) \leq \eta \cdot n \cdot 2C\nu_{m-1} + N \leq (2C\eta+N)n\nu_{m-1} = (2C\eta+N)n^2\nu_m^2 .$$

By the above and assumption (3), for $1 \leq m \leq j$, simple calculations reveal that

$$\sum_{k\in I_m(x)} h_{nk}(x) \leq \frac{1}{2j} ,$$

so that

$$\sum_{k\in I_{j+1}(x)} h_{nk}(x) \geq \frac{1}{2} ,$$

and hence

$$\sum_{k\in I_{j+1}(x)} h_{nk}^2(x) \geq \frac{1}{4\ \text{card}\big(I_{j+1}(x)\big)} \geq \frac{n^{-2^{-j}}}{8C\eta+N} \geq A_\alpha n^{-\alpha}$$

with

$$A_\alpha = \frac{1}{8C\eta+4N} .$$

Part (ii): The proof of part (ii) is much the same as that of part (i). For each n s.t. $\alpha_n < 1$, choose $j_n \in Z^+$ s.t. $2^{-j_n} \le \alpha_n < 2^{-j_n+1}$. Then $j_n = O(|\ln\alpha_n|)$ as $n \to \infty$ (of course $\alpha_n < 1$ for n large enough). Also let $C_n = \max\{1, 2j_n(2\eta+N)M\}$, so $C_n \le K|\ln\alpha_n|$ for some $K > 0$ independent of n. Let $m \ge n_0$ s.t. $n \ge m \Rightarrow C_n n^{-(1/2)} < \varepsilon - \mu$ (possible since $n^{-(1/2)}|\ln\alpha_n| \to 0$). Now, as in the proof of (i), if $n \ge m$ and $x \in I_\varepsilon$, then

$$\sum_{k=1}^{n} h_{nk}^2(x) \ge \left(\frac{1}{8C_n\eta + 4N}\right)n^{-\alpha_n} .$$

Let $n_3 \ge m$ s.t. $n \ge n_3 \Rightarrow \alpha_n < 1$ and $C_n\eta > 4N$ (possible since $\alpha_n \to 0 \Rightarrow j_n \to \infty \Rightarrow C_n \to \infty$) and then $n \ge n_3 \Rightarrow \sum_{k=1}^{n} h_{nk}^2(x) \ge \frac{n^{-\alpha_n}}{9C_n\eta} \ge \frac{1}{9K\eta}(n^{\alpha_n}|\ln\alpha_n|)^{-1}$. Hence (ii) is established with

$$A = \frac{1}{9K\eta} .$$

Proof of Lemma 4.2: This follows immediately from Lemma 4.1 by setting $\alpha_n = (\ln n)^{-1}$, which yields, for $n \ge n_3$,

$$\frac{\sum_{k=1}^{n} h_{nk}^2(x)(x-x_{nk})^2}{\sum_{k=1}^{n} h_{nk}^2(x)} \le \frac{n^{\alpha_n}|\ln\alpha_n|\, Mn^{-2}\sum_{k=1}^{n} h_{nk}(x)}{A}$$

$$\le \frac{(\ln \ln n)\cdot MC_2}{n^2 A} ,$$

so the lemma is established with $B = \frac{MC_2}{A}$ and $n_4 = n_3$.

Proof of Theorem 4.1: By lemma 4.2, we have

$$\|L_n((x-t)^2;x)\|_\infty \le \frac{B \ln \ln n}{n^2} .$$

Theorem 4.1 follows immediately from this inequality and Corollary 2.1.

A few easy modifications in the proofs of the above lemmas verify the following extension (see Theorem 2.2 and the following remark).

Lemma 4.3: If p is an integer, $p \geq 2$, then there exists a constant $B_p > 0$ and an integer m_p s.t. whenever $n \geq m_p$, $x \in I_\varepsilon$, we have

$$\frac{\sum_{k=1}^{n} h_{nk}^{p}(x)(x-x_{nk})^{2(p-1)}}{\sum_{k=1}^{n} h_{nk}^{p}(x)} \leq B_p\left(\frac{\ell n\ \ell n\ n}{n^2}\right)^{p-1}.$$

Using this lemma, one can easily prove a theorem which actually implies Theorem 4.1 with the choice $p = 2$:

Theorem 4.2. Let p be any integer with $p \geq 2$, and let points $-1 \leq x_{nn} < \cdots < x_{n1} \leq 1$ and functions $h_{n1}(x),\cdots,h_{nn}(x)$ be given for every $n = 1,2,\cdots$; suppose there exist constants $\eta > 0$, $M > 0$, $\varepsilon > \mu > 0$, $C_2 > C_1 > 0$, and an integer n_1 s.t., for $x \in I_\varepsilon$, assumptions (1) thru (4) in the statement of Theorem 4.1 are satisfied. Define

$$L_{n,p}f(x) = \frac{\sum_{k=1}^{n} h_{nk}^{p}(x)f(x_{nk})}{\sum_{k=1}^{n} h_{nk}^{p}(x)}$$

for $f \in C[-1,1]$, $x \in I_\varepsilon$.

Then, if $\|\cdot\|_\infty$ denotes the sup norm on I_ε, there exists $C_p > 0$ independent of n and f s.t.

$$\|L_{n,p}f-f\|_\infty \leq C_p\omega_f\left(\frac{\sqrt{\ell n\ \ell n\ n}}{n}\right).$$

The following result will assist in demonstrating the applicability of Theorems 4.1 and 4.2 to several examples:

Theorem 4.3. Let $p_n(x)$ be one of the classical polynomials $T_n(x)$, $P_n(x)$, $U_n(x)$, $P'_{n+1}(x)$, or $(1-x^2)P'_{n-1}(x)$, and let p_n have zeros $-1 \le x_{nn} < \cdots < x_{n1} \le 1$, for $n = 1,2,\cdots$. Then for every $\mu > 0$ there exists $\eta > 0$ s.t. the x_{nk} are η-dispersed in $[-1+\mu,1-\mu]$.

Proof: It is clearly enough to prove this for the choices $p_n = T_n$ or P_n, since the zeros of the other choices alternate with the Tchebycheff or Legendre zeros, by the Mean Value Theorem. T_n and P_n are examples of the classical ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$, with $\alpha = -\frac{1}{2}$, 0 respectively. Therefore a separation theorem (Szegö [10], pp. 122) due to Stieltjes can be applied, and states that if $x_{nk} = \cos \theta_{nk}$, $k = 1,\cdots,n$, are the zeros of $P_n^{(\alpha,\alpha)}(x)$ with $-\frac{1}{2} \le \alpha \le \frac{1}{2}$, then

$$\left(k - \frac{1}{2}\right)\frac{\pi}{n} \le \theta_{nk} \le k\,\frac{\pi}{n+1}\,, \quad k = 1,2,\cdots,[\tfrac{n}{2}] \quad (\text{i.e.,} \quad \theta_{nk} < 0);$$

for the k for which $\theta_{nk} > 0$, a similar result clearly occurs by the symmetry of the zeros about the point $\theta = \frac{\pi}{2}$.

Let $\mu > 0$; then there exists some $\gamma > 0$ s.t. $x,t \in [-1+\mu,1-\mu]$ $\Rightarrow |x-t| \ge \gamma|\arccos x - \arccos t|$. $\therefore$ if $1 \le k \le [\frac{n}{2}] - 1$ and x_{nk}, $x_{nk+1} \in [-1+\mu,1-\mu]$, then

$$|x_{nk} - x_{nk+1}| \ge \gamma|\arccos x_{nk} - \arccos x_{nk+1}| = \gamma|\theta_{nk}-\theta_{nk+1}|$$
$$\ge \gamma\left(\frac{\left(k + \frac{1}{2}\right)}{n} - \frac{k\pi}{n+1}\right) > \frac{\gamma\pi}{2n}\,,$$

and similarly for pairs x_{nk}, x_{nk+1} on the other side of zero $(k = [\frac{n+3}{2}],\cdots,n)$. Two cases are considered (n even and n odd) for the zeros at or next to the point zero. If n is even, let $k = \frac{n}{2} = [\frac{n}{2}]$,

and then

$$\theta_{nk} \le \frac{k\pi}{n+1} < \frac{(k+1)\pi}{n+1} \le \theta_{nk+1} \qquad \text{(by symmetry)}$$

$$\Rightarrow |x_{nk}-x_{nk+1}| > \gamma\left(\frac{\pi}{n+1}\right) > \frac{\gamma\pi}{2n}\ .$$

If n is odd, let $k = \frac{n-1}{2} = [\frac{n}{2}]$, and then

$$\theta_{nk} \le \frac{k\pi}{n+1} < \frac{\pi}{2} = \theta_{nk+1} < \frac{(k+2)\pi}{n+1} \le \theta_{nk+2},\quad \frac{\pi}{2} = \frac{(k+1)\pi}{n+1}\ ,$$

so that $|x_{nk}-x_{nk+1}| = |x_{nk+1}-x_{nk+2}| \ge \gamma\,\frac{\pi}{n+1} > \frac{\gamma\pi}{2n}$.

Therefore the distance between any two consecutive zeros x_{nk} and x_{nk+1} in $[-1+\mu,1-\mu]$ is at least $\frac{\gamma\pi}{2n}$. Therefore if I is any subinterval of $[-1+\mu,1-\mu]$ of length $\ell(I)$ then

$$\text{card}\,\{k \mid x_{nk} \in I\} \le \frac{2n}{\gamma\pi}\cdot \ell(I) + 1,$$

so the $\{x_{nk}\}$ are η-dispersed in $[-1+\mu,1-\mu]$ where $\eta = \frac{2}{\gamma\pi}$.

<u>Remark</u>. The classical ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ satisfy a differential equation of the form

$$(1-x^2)y'' - \nu xy' + n(n + \nu - 1)y = 0,$$

where $\nu = 2\alpha + 2$. Note that if p_n satisfies such an equation and has roots x_{nk} $(k = 1,\cdots,n)$, and if $q_n(x) = (1-x^2)p_n(x)$, then

$$\frac{p_n''(x_{nk})}{p_n'(x_{nk})} = \frac{\nu\, x_{nk}}{1-x_{nk}^2}\ ,$$

and

$$\frac{q_n''(x_{nk})}{q_n'(x_{nk})} = \frac{(\nu-4)x_{nk}}{1-x_{nk}^2}\ .$$

Therefore

$$\left(1 - (x-x_{nk}) \frac{p_n''(x_{nk})}{p_n'(x_{nk})}\right) = \frac{1 - \nu x_{nk}x + (\nu-1)x_{nk}^2}{1 - x_{nk}^2}$$

and

$$\left(1 - (x-x_{nk}) \frac{q_n''(x_{nk})}{q_n'(x_{nk})}\right) = \frac{1 - (\nu-4)x_{nk}x + (\nu-5)x_{nk}^2}{1 - x_{nk}^2}$$

In each of the following examples, $-1 \leq x_{nn} < \cdots < x_{n1} \leq 1$ will denote the zeros of $p_n(x)$. It should be noted that Theorem 4.3 serves to establish in each case that the x_{nk} are η-dispersed in $[-1+\mu, 1-\mu]$ for each $\mu > 0$ and appropriate η. It remains therefore to verify assumptions (1)-(3) of Theorem 4.1 in each case.

<u>Example 1</u>:

$$p_n(x) = T_n(x),\ h_{nk}(x) = \frac{(1-x_{nk}x)T_n^2(x)}{(1-x_{nk}^2)(x-x_{nk})^2\left(T_n'(x_{nk})\right)^2} = \frac{(1-x_{nk}x)T_n^2(x)}{n^2(x-x_{nk})^2} .$$

Hence
$$0 \leq h_{nk}(x) \leq \frac{2}{n^2(x-x_{nk})^2}$$

and $\sum_{k=1}^{n} h_{nk}(x) \equiv 1$ (see the remarks at the end of Chapter 3, as well as the definition of H-F interpolation.) In this case, one can choose any $\varepsilon > \mu > 0$.

<u>Example 2</u>: $p_n(x) = P_n(x)$, $h_{nk}(x) = \dfrac{(1 - 2x_{nk}x + x_{nk}^2)P_n^2(x)}{(1-x_{nk}^2)(x-x_{nk})^2\left(P_n'(x_{nk})\right)^2}$.

Once again $h_{nk}(x) \geq 0$ on $[-1,1]$ and $\sum_{k=1}^{n} h_{nk}(x) \equiv 1$. The property that for $\varepsilon > 0$ there exists an M s.t.

$$h_{nk}(x) \leq \frac{M}{n^2(x-x_{nk})^2}$$

for all $x \in I$, and all n, k, requires certain estimates:

$$|P_n(x)| = 0\left(n^{-(1/2)}(1-x^2)^{-(1/4)}\right), \quad \text{all } n$$

$$1 - x_{nk}^2 > \begin{cases} k^2 n^{-2} & , \text{ if } k \le \frac{n}{2}, \quad \text{all } n,k \\ (n+1-k)^2 n^{-2}, & \text{if } k > \frac{n}{2} \end{cases}$$

$$a_2 k^{-(3/2)} n^2 \le |P_n'(x_{nk})| \le a_1 k^{-(3/2)} n^2, \quad \text{if } k \le \frac{n}{2}$$

$$a_2 (n+1-k)^{-(3/2)} n^2 \le |P_n'(x_{nk})| \le a_1 (n+1-k)^{-(3/2)} n^2, \quad \text{if } k > \frac{n}{2},$$

with constants $a_1 > a_2 > 0$. These estimates may be found in Szegö [10; pp. 122, 165, 238].

Let $\varepsilon > 0$ and restrict $x \in I_\varepsilon$. Denoting $j = \min\{k,n+1-k\}$, it follows from the above inequalities that there exists $K > 0$ independent of n,k,x s.t.

$$\begin{aligned} h_{nk}(x) &\le \frac{4K}{j^2 n^{-2} \; n\sqrt{1-x^2} \; j^{-3} n^4 (x-x_{nk})^2} \\ &= 0\left(\frac{1}{n^2 (x-x_{nk})^2}\right). \end{aligned}$$

<u>Example 3</u>: $p_n(x) = U_n(x)$, $\quad h_{nk}(x) = \dfrac{(1-3x_{nk}x + 2x_{nk}^2)U_n^2(x)}{(1-x_{nk}^2)(x-x_{nk})^2 \left(U_n'(x_{nk})\right)^2}$.

In this case it is necessary to choose $\varepsilon \ge \frac{3-2\sqrt{2}}{3}$ to ensure that $x \in I_\varepsilon \Rightarrow h_{nk}(x) \ge 0$. We do have $\sum_{k=1}^{n} h_{nk}(x) \equiv 1$.

Since $|U_n'(x_{nk})| = \dfrac{n+1}{1-x_{nk}^2}$ and $U_n^2(x) \le \sin^{-2}(\text{arc cos } x) = 0(1)$

on I_ε, it follows that

$$h_{nk}(x) \le \frac{6(1-x_{nk}^2) \cdot 0(1)}{(n+1)^2(x-x_{nk})^2} = 0\left(\frac{1}{n^2(x-x_{nk}^2)}\right) .$$

<u>Example 4</u>: $p_n(x) = P'_{n+1}(x)$, $h_{nk}(x) = \dfrac{(1-4x_{nk}x + 3x_{nk}^2)\left(P'_{n+1}(x)\right)^2}{(1-x_{nk}^2)(x-x_{nk})^2\left(P''_{n+1}(x_{nk})\right)^2}$.

If $\varepsilon = \dfrac{2-\sqrt{3}}{2}$, then $x \in I_\varepsilon \Rightarrow h_{nk}(x) \ge 0$ for all n,k.

Also, again, $\sum\limits_{k=1}^{n} h_{nk}(x) \equiv 1$.

Two inequalities to be used can be found in Balázs and Turan [2] or Szegö [10]:

$$|P'_{n+1}(x)| \le \sqrt{2(n+2)}\ (1-x^2)^{-(3/4)}$$

and

$$|P_{n+1}(x_{nk})| \ge (8\pi j)^{-(1/2)},$$

where $j = \min\{k,n+2-k\}$.

Furthermore, the differential equation

$$(1-x^2)P''_{n+1}(x) - 2xP'_{n+1}(x) + (n+1)(n+2)P_{n+1}(x) = 0$$

yields

$$(1-x_{nk}^2)P''_{n+1}(x_{nk}) = -(n+1)(n+2)P_{n+1}(x_{nk}),$$

so that

$$h_{nk}(x) \le \frac{8(1-x_{nk}^2) \cdot 2(n+2)(1-x^2)^{-(3/2)} \cdot 8\pi j}{(x-x_{nk})^2(n+1)^2(n+2)^2}$$

$$= 0\left(\frac{1}{n^2(x-x_{nk})^2}\right) .$$

<u>Example 5</u>: $p_n(x) = (1-x^2)P'_{n+1}(x)$, $h_{nk}(x) = \dfrac{(1-x_{nk}^2)\left((1-x^2)P'_{n+1}(x)\right)^2}{(1-x_{nk}^2)(x-x_{nk})^2\left(p'_n(x_{nk})\right)^2}$.

In this case the $\{x_{nk}\}$ are taken to be the zeros of $P'_{n+1}(x)$ (the other roots, $x = \pm 1$, of $p_n(x)$ are disregarded for simplicity).

Letting

$$h_{no}(x) = \left(1 - (x-1)\frac{p''_n(1)}{p'_n(1)}\right)\frac{p_n^2(x)}{(x-1)^2(p'_n(1))^2}$$

and

$$h_{n,n+1}(x) = \left(1 - (x+1)\frac{p''_n(-1)}{p'_n(-1)}\right)\frac{p_n^2(x)}{(x+1)^2(p'_n(p'_n(-1))^2} ,$$

one gets

$$\sum_{k=o}^{n+1} h_{nk}(x) = 1.$$

Now, $p'_n(x) = -(n^2+n)P_n(x)$ (from the differential equation) and $p''_n(x) = -(n^2+n)P'_n(x)$, so that

$$|P'_n(\pm 1)| = \tfrac{1}{2}(n^2+n), \qquad \left|\frac{p''_n(\pm 1)}{p'_n(\pm 1)}\right| = \frac{n^2+n}{2} .$$

Therefore if $k = 0$ or $n+1$, $\varepsilon > 0$, and $x \in I_\varepsilon$,

$$|h_{nk}(x)| \le \left(1 + \frac{2(n^2+n)}{2}\right)\frac{\left((1-x^2)P'_n(x)\right)^2}{\varepsilon^2(n^2+n)^2}$$

$$= 0\left(\frac{1}{n^2}\right)\left((1-x^2)P'_n(x)\right)^2$$

$$= 0\left(\frac{1}{n}\right) , \quad \text{using} \quad |P'_n(x)| < \sqrt{2(n+1)}\,(1-x^2)^{-(3/4)}.$$

Therefore, for n large enough, $|h_{no}(x) + h_{nn+1}(x)| < \frac{1}{2}$ for all $x \in I_\varepsilon$, so that

$$\frac{1}{2} \le \sum_{k=1}^{n} h_{nk}(x) \le \frac{3}{2} .$$

Finally, if $1 \le k \le n$ and $x \in I_\varepsilon$, then

$$h_{nk}(x) = \frac{(1-x_{nk}^2)(1-x^2)^2 \left(P_n'(x)\right)^2}{(1-x_{nk}^2)(x-x_{nk})^2 (n^2+n)^2 P_n^2(x_{nk})} \qquad (\ge 0, \quad \text{clearly})$$

$$= \frac{\left((1-x^2)^{3/4} P_n'(x)\right)^2 \sqrt{1-x^2}}{(n^2+n)^2 P_n^2(x_{nk})(x-x_{nk})^2} \le \frac{2(n+1)\sqrt{1-x^2}\,(8\pi j)}{(n^2+n)^2 (x-x_{nk})^2}$$

where $j = \min\{k, n+1-k\}$ (using the two inequalities quoted in Example 4)

$$= O\left(\frac{1}{n^2 (x-x_{nk})^2}\right) .$$

From these examples it is evident that Theorems 4.1 and 4.2 apply to a large number of cases.

CHAPTER 5

ACHIEVING A BETTER DEGREE OF APPROXIMATION

In this chapter, three approximation theorems which are related to the results of Chapter 4 will be stated and proven. The special properties of each sequence of operators will be used to attain good degrees of approximation.

As in the previous chapters, ω_f will denote the modulus of continuity of $f \in C[-1,1]$.

<u>Theorem 5.1</u>. Let $x_{n1},\cdots,x_{nn}$ $(-1 < x_{nn} < \cdots < x_{n1} < 1)$ be the zeros of $T_n(x)$ and define $L_n : C[-1,1] \to C[-1,1]$ by

$$L_n f(x) = \frac{\sum_{k=1}^{n} m_{nk}(x) f(x_{nk})}{\sum_{k=1}^{n} m_{nk}(x)},$$

where

$$m_{nk}(x) = \frac{T_n^4(x)}{n^4(x-x_{nk})^4}\left[(1-x^2)(1-x_{nk}^2) + \frac{1}{2}(x-x_{nk})^2\right].$$

Then L_n is a positive linear interpolation operator, and if $||\cdot||_\infty$ denotes the sup norm on $[-1,1]$, we have

$$||L_n f - f||_\infty \le 4\omega_f\left(\frac{1}{n}\right).$$

<u>Proof of Theorem 5.1</u>: In [6] there appears the following representation of the basic polynomial $\lambda_{nk}(x)$ of the extended H-F polynomial interpolation process with Tchebycheff nodes (see Chapter 3):

$$\lambda_{nk}(x) = \frac{T_n^4(x)}{n^4(x-x_{nk})^4}\left\{(1-x_{nk}x)^2 + (x-x_{nk})^2\,\frac{2(n^2-1)}{3}\,(1-x_{nk}x) - \frac{1}{2}\,x_{nk}x\,\right\}$$

which can be rewritten:

$$\lambda_{nk}(x) = \frac{T_n^4(x)}{n^4(x-x_{nk})^4}\left\{(1-x^2)(1-x_{nk}^2) + \frac{1}{2}(x-x_{nk})^2 + \left(\frac{4n^2-1}{6}\right)(x-x_{nk})^2(1-x_{nk}x)\right\}$$

so that

$$1 \equiv \sum_{k=1}^{n}\lambda_{nk}(x) = \sum_{k=1}^{n} m_{nk}(x) + \frac{(4n^2-1)}{6n^2}\,T_n^2(x)\sum_{k=1}^{n}\frac{T_n^2(x)(1-x_{nk}x)}{n^2(x-x_{nk})^2}\,.$$

Now, since $\dfrac{T_n^2(x)(1-x_{nk}x)}{n^2(x-x_{nk})^2}$, $k = 1,\cdots,n$, are the basic polynomials for the H-F process on the nodes x_{nk}, one obtains $\sum\limits_{k=1}^{n} m_{nk}(x) \geq \frac{1}{3}$,

Also

$$\sum_{k=1}^{n} m_{nk}(x)(x-x_{nk})^2 = \sum_{k=1}^{n}\frac{T_n^4(x)}{n^4(x-x_{nk})^2}\left(1-x_{nk}x-\tfrac{1}{2}x^2(1-x_{nk}^2)-\tfrac{1}{2}x_{nk}^2(1-x^2)\right)$$

$$\leq \frac{1}{n^2}\sum_{k=1}^{n}\frac{T_n^2(x)(1-x_{nk}x)}{n^2(x-x_{nk})^2} = \frac{1}{n^2}\quad,$$

and so the result follows from Corollary 2.1.

Theorems 5.2 and 5.3 deal with approximation by interpolation operators having as nodes the zeros of $U_n(x)$ and of $P_n(x)$.

<u>Theorem 5.2</u>: Let $x_{n1},\cdots,x_{nn}$ $(-1 < x_{nn} < \cdots < x_{n1} < 1)$ be the zeros of $U_n(x)$ and let $\varepsilon > 0$. Define $L_n : C[-1,1] \to C(I_\varepsilon)$ by:

$$L_n f(x) = \frac{\sum\limits_{k=1}^{n} m_{nk}(x) f(x_{nk})}{\sum\limits_{k=1}^{n} m_{nk}(x)}\,,$$

where

$$m_{nk}(x) = \left(\frac{U_n(x)}{(x-x_{nk})U_n'(x_{nk})}\right)^4 .$$

Let $\|\cdot\|_\infty$ denote the sup norm on I_ε. Then there exists a constant $A > 0$ and an integer n_1 s.t. for every $f \in C[-1,1]$ and every $n \geq n_1$,

$$\|L_n f - f\|_\infty \leq A\omega_f\left(\frac{1}{n}\right) .$$

<u>Remark</u>. The fact that the error estimate in Theorem 5.2 holds only on I_ε, and not on $[-1,1]$, as in Theorem 5.1, is not really a restriction, for if $f \in C[-1,1]$ one can approximate f on the whole interval with a modest amount of juggling:

$$\text{Define}\quad g(x) = \begin{cases} f(-1) , & x < -1+\varepsilon \\ f\left(\frac{x}{1-\varepsilon}\right), & x \in I_\varepsilon \\ f(1) , & x > 1-\varepsilon \end{cases}$$

so that $\omega_g(\delta) = \omega_f\left((1-\varepsilon)^{-1}\delta\right) \leq \left((1-\varepsilon)^{-1} + 1\right)\omega_f(\delta)$. By Theorem 5.2, if $x \in I_\varepsilon$,

$$|L_n g(x) - g(x)| \leq A\omega_g\left(\frac{1}{n}\right) \leq A\left((1-\varepsilon)^{-1} + 1\right)\omega_f\left(\frac{1}{n}\right) ,$$

and, letting $t = \frac{x}{1-\varepsilon}$, one gets that, for $t \in [-1,1]$,

$$|L_n g\left((1-\varepsilon)t\right) - f(t)| \leq A\left((1-\varepsilon)^{-1}+1\right)\omega_f\left(\frac{1}{n}\right) .$$

Thus, one obtains, by changes of variables, an approximation to f on all of $[-1,1]$.

The proof of Theorem 5.2 will require some knowledge of the H-F and extended H-F polynomial interpolation processes on the nodes x_{nk}.

If, as in Chapter 3, one denotes by $h_{nk}(x)$ and $\lambda_{nk}(x)$ the basic polynomials of, respectively, the H-F and extended H-F process on the nodes x_{nk}, then one has the representations:

$$h_{nk}(x) = \frac{(1-3x_{nk}x + 2x_{nk}^2)}{1 - x_{nk}^2} \left(\frac{U_n(x)}{(x-x_{nk})U_n'(x_{nk})}\right)^2$$

and

$$\lambda_{nk}(x) = \left(1 + C_1(x-x_{nk}) + C_2(x-x_{nk})^2 + C_3(x-x_{nk})^3\right)\left[\frac{U_n(x)}{(x-x_{nk})U_n'(x_{nk})}\right]^4$$

where (see Sharma and Tzimbalario [8])

$$C_1 = \frac{-6x_{nk}}{1-x_{nk}^2}$$

$$C_2 = \frac{25x_{nk}^2}{2(1-x_{nk}^2)^2} - 2\left(\frac{1}{1-x_{nk}^2}\right) + \frac{2N}{3(1-x_{nk}^2)}$$

$$C_3 = \frac{15x_{nk} - 35x_{nk}^3}{2(1-x_{nk}^2)^3} - \frac{10N\,x_{nk}}{3(1-x_{nk}^2)^2} \qquad (N = n^2 + 2n)$$

Some other facts will be needed:

(1) $$|U_n'(x_{nk})| = \frac{n+1}{1-x_{nk}^2}$$

(2) If $0 < \varepsilon < 1$ and $M_\varepsilon = \dfrac{1}{\sin \operatorname{arc}\cos(1-\varepsilon)} < \infty$, then, for every $x \in I_\varepsilon$, $|U_n(x)| \leq M_\varepsilon$.

(3) If $\ell_{nk}(x) = \dfrac{U_n(x)}{(x-x_{nk})U_n'(x_{nk})}$ for all n,k, and $K_\varepsilon = \max(M_\varepsilon, 2)$, then there exists m_o s.t. if $n \geq m_o$, $1 \leq k \leq n$, and $x \in I_\varepsilon$, then $|\ell_{nk}(x)| \leq K_\varepsilon$. To see this is not difficult. First, let m_o be s.t. $n \geq m_o \Rightarrow \frac{1}{n+1} < \sqrt{6}\, M_\varepsilon n^{-(1/2)} \leq \frac{\varepsilon}{12}$. Let n,k,x be fixed s.t. $x \in I_\varepsilon$, $n \geq m_o$, $1 \leq k \leq n$, and put $M = \sqrt{6}\, M_\varepsilon\, n^{-(1/2)}$. Either $|x-x_{nk}| \geq \frac{1}{n+1}$

or $|x-x_{nk}| < \frac{1}{n+1}$. In the former case, one has

$$|\ell_{nk}(x)| = \left|\frac{U_n(x)(1-x_{nk}^2)}{(x-x_{nk})(n+1)}\right| \le M_\varepsilon .$$

In case $|x-x_{nk}| < \frac{1}{n+1}$, we have

$$\begin{aligned}
\sum_{|x-x_{nj}|\le M} h_{nj}(x) &\le 1 + \sum_{|x-x_{nj}|>M} |h_{nj}(x)| \\
&= 1 + \sum_{|x-x_{nj}|>M} \left|\frac{(1-3x_{nj}x + 2x_{nj}^2)(1-x_{nj}^2)U_n^2(x)}{(n+1)^2(x-x_{nj})^2}\right| \\
&\le 1 + \sum_{|x-x_{nj}|>M} \frac{6\cdot M_\varepsilon^2\cdot n}{6M^2(n+1)^2} < 2 .
\end{aligned}$$

Now, if $|x-x_{nj}| \le M \le \frac{\varepsilon}{12}$, $(1-x_{nk}^2) > (1-x_{nk}) > \frac{\varepsilon}{2}$, so

$$\begin{aligned}
h_{nj} &= \left(1 - \frac{3x_{nj}(x-x_{nj})}{1-x_{nj}^2}\right)\ell_{nj}^2(x) \\
&\ge \left(1 - \frac{6|x_{nj}|M}{\varepsilon}\right)\ell_{nj}^2(x) \\
&\ge \frac{1}{2}\ell_{nj}^2(x) > 0.
\end{aligned}$$

Therefore $\ell_{nk}^2(x) \le 2h_{nk}(x) \le 2\cdot \sum_{|x-x_{nj}|\le M} h_{nj}(x) < 4$ so $\ell_{nk}(x) < 2 \le K_\varepsilon$.

Since the first finitely many $\ell_{nk}(x)$ are also uniformly bounded in I_ε,

$$K = \sup\{|\ell_{nk}(x)| \,\big|\, n = 1,2,\cdots,\ 1 \le k \le n,\ x \in I_\varepsilon\} < \infty.$$

(4) Let $p_{nk}(x)$, $1 \le k \le n$, $n = 1,2,\cdots$ be given s.t.

$$B = \sup\{|p_{nk}(x)| \,\big|\, 1 \le k \le n,\ x \in I_\varepsilon\} < \infty .$$

Let $\eta > 0$ s.t. the x_{nk} are η-dispersed in I_ε ($\varepsilon > 0$ fixed).

Then

$$\sum_{k=1}^{n} \frac{p_{nk}(x)(x-x_{nk})\ell_{nk}^2(x)}{1-x_{nk}^2} = 0\left(\frac{1}{\sqrt{n}}\right)$$

where the "0" constant is independent of $x \in I_\varepsilon$. The proof of this is elementary:

$$\sum_{|x-x_{nk}|<\frac{1}{\sqrt{n}}} \frac{|p_{nk}(x)||x-x_{nk}|\ell_{nk}^2(x)}{1-x_{nk}^2}$$

$$\leq B \sum_{|x-x_{nk}|<\frac{1}{\sqrt{n}}} \frac{|U_n(x)||\ell_{nk}(x)|}{n+1} \leq B \cdot 0(\sqrt{n}) \cdot \frac{M_\varepsilon \cdot K}{n+1}$$

$$= 0\left(\frac{1}{\sqrt{n}}\right),$$

and

$$\sum_{|x-x_{nk}|\geq\frac{1}{\sqrt{n}}} |p_{nk}(x)| \frac{|x-x_{nk}|\ell_{nk}^2(x)}{1-x_{nk}^2}$$

$$\leq B \cdot \sum_{|x-x_{nk}|\geq\frac{1}{\sqrt{n}}} \frac{(1-x_{nk}^2)U_n^2(x)}{(n+1)^2|x-x_{nk}|} \leq \frac{B \cdot n \cdot M_\varepsilon^2 \cdot \sqrt{n}}{(n+1)^2} = 0\left(\frac{1}{\sqrt{n}}\right),$$

so

$$\sum_{k=1}^{n} p_{nk}(x) \frac{(x-x_{nk})}{1-x_{nk}^2} \ell_{nk}^2(x) = 0\left(\frac{1}{\sqrt{n}}\right),$$

as claimed.

<u>Lemma 5.1</u>: $\sum_{k=1}^{n} \ell_{nk}^2(x) = 1 + 0\left(\frac{1}{\sqrt{n}}\right)$, as $n \to \infty$, where the "0" constant is independent of $x \in I_\varepsilon$.

Proof:

$$1 = \sum_{k=1}^{n} h_{nk}(x) = \sum_{k=1}^{n} \ell_{nk}^2(x) + \sum_{k=1}^{n} \frac{3x_{nk}(x-x_{nk})}{1-x_{nk}^2}\ell_{nk}^2(x)$$

$$= \sum_{k=1}^{n} \ell_{nk}^2(x) + 0\left(\frac{1}{\sqrt{n}}\right) \quad \text{by (4) above.}$$

Lemma 5.2:
$$\sum_{k=1}^{n}\left[\ell_{nk}^4(x) + \frac{2N\, U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x)}{3(n+1)^2}\right] = 1 + 0\left(\frac{1}{\sqrt{n}}\right)$$

uniformly for $x \in I_\varepsilon$, as $n \to \infty$.

Proof:

$$\sum_{k=1}^{n} \lambda_{nk}(x) = 1, \quad \text{and}$$

$$\lambda_{nk}(x) = \left(1 + C_1(x-x_{nk}) + C_2(x-x_{nk})^2 + C_3(x-x_{nk})^3\right)\ell_{nk}^4(x),$$

where

$$C_1 = \frac{-6x_{nk}}{1-x_{nk}^2}, \qquad C_2 = \frac{29x_{nk}^2-4}{2(1-x_{nk}^2)^2} + \frac{2N}{3(1-x_{nk}^2)},$$

$$C_3 = \frac{15x_{nk}-35x_{nk}^3}{2(1-x_{nk}^2)^3} - \frac{10N\,x_{nk}}{3(1-x_{nk}^2)^2},$$

so
$$\sum_{k=1}^{n} \lambda_{nk}(x) = \sum_{k=1}^{n}\left[\ell_{nk}^4(x) + \frac{2N(x-x_{nk})^2}{3(1-x_{nk}^2)}\ell_{nk}^4(x)\right]$$

$$+ \sum_{k=1}^{n} \frac{-6x_{nk}(x-x_{nk})}{1-x_{nk}^2}\ell_{nk}^4(x) \qquad (= \textstyle\sum_1)$$

$$+ \sum_{k=1}^{n} \frac{(29x_{nk}^2-4)(x-x_{nk})^2}{2(1-x_{nk}^2)^2}\ell_{nk}^4(x) \qquad (= \textstyle\sum_2)$$

$$+ \sum_{k=1}^{n} \frac{(15x_{nk}-35x_{nk}^3)(x-x_{nk})^3}{2(1-x_{nk}^2)^3}\ell_{nk}^4(x) \quad (= \textstyle\sum_3)$$

$$+\sum_{k=1}^{n} \frac{-10Nx_{nk}(x-x_{nk})^3}{3(1-x_{nk}^2)^2} \ell_{nk}^4(x). \qquad (= \sum_4)$$

Now,

$$|\textstyle\sum_1| = |\displaystyle\sum_{k=1}^{n} \frac{\left(x_{nk}\ell_{nk}^2(x)\right)(x-x_{nk})}{1-x_{nk}^2} \ell_{nk}^2(x)| = 0\left(\frac{1}{\sqrt{n}}\right)$$

by remarks (3) and (4).

$$|\textstyle\sum_2| = |\displaystyle\sum_{k=1}^{n} \frac{(29x_{nk}^2-4)U_n^2(x)\ell_{nk}^2(x)}{2(n+1)^2}| = 0\left(\frac{1}{n}\right)$$

by remarks (2) and (3).

$$|\textstyle\sum_3| = |\displaystyle\sum_{k=1}^{n} \frac{(15x_{nk}-35x_{nk}^3)U_n^2(x)(x-x_{nk})}{2(n+1)^2(1-x_{nk}^2)} \ell_{nk}^2(x)| = 0\left(n^{-(5/2)}\right)$$

by remarks (2) and (4).

$$|\textstyle\sum_4| = |\displaystyle\sum_{k=1}^{n} \frac{10Nx_{nk}U_n^2(x)(x-x_{nk})(1-x_{nk}^2)}{3(n+1)^2(1-x_{nk}^2)} \ell_{nk}^2(x)| = 0\left(\frac{1}{\sqrt{n}}\right)$$

since $\frac{N}{(n+1)^2} < 1$, and also by remarks (2) and (4). Finally,

$$\frac{2N(x-x_{nk})^2}{3(1-x_{nk}^2)} \ell_{nk}^4(x) = \frac{2N\, U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x)}{3(n+1)^2},$$

so the lemma is established.

<u>Lemma 5.3</u>: $$\sum_{k=1}^{n} \frac{2N\, U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x)}{3(n+1)^2} \le \frac{2}{3} + 0\left(n^{-(1/3)}\right)$$

uniformly on $x \in I_\varepsilon$, as $n \to \infty$.

Proof:

$$\sum_{k=1}^{n} U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x)$$

$$= \sum_{|x-x_{nk}|\geq n^{-(1/3)}} U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x) \qquad (=\textstyle\sum_1)$$

$$+ \sum_{|x-x_{nk}|<n^{-(1/3)}} U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x) \qquad (=\textstyle\sum_2).$$

Now,

$$|\textstyle\sum_1| = \left| \displaystyle\sum_{|x-x_{nk}|\geq n^{-(1/3)}} \frac{U_n^4(x)(1-x_{nk}^2)^3}{(x-x_{nk})^2(n+1)^2} \right|$$

$$\leq M_\varepsilon^4 \cdot n \cdot (n+1)^{-2} \cdot n^{2/3} = O\left(n^{-(1/3)}\right)$$ (see remarks preceding Lemma 5.1).

Notice that $U_n^2(x)(1-x_{nk}^2) = \dfrac{\sin^2((n+1)\arccos x)(1-x_{nk}^2)}{1-x^2}$

$$\leq \frac{1-x_{nk}^2}{1-x^2} = 1 + O(|x-x_{nk}|),$$

so

$$\textstyle\sum_2 = \displaystyle\sum_{|x-x_{nk}|<n^{-(1/3)}} U_n^2(x)(1-x_{nk}^2)\ell_{nk}^2(x)$$

$$\leq \left(1 + O(n^{-(1/3)})\right) \sum_{k=1}^{n} \ell_{nk}^2(x)$$

$$\leq \left(1 + O(n^{-(1/3)})\right)\left(1 + O(n^{-(1/2)})\right) = 1 + O(n^{-(1/3)})$$

(see Lemma 5.1), and the lemma is established (since $\dfrac{2N}{3(n+1)^2} < \dfrac{2}{3}$).

Lemma 5.4:

$$\sum_{k=1}^{n} \ell_{nk}^4(x) \geq \frac{1}{3} + O(n^{-(1/3)}), \quad \text{as} \quad n \to \infty,$$

uniformly for $x \in I_\varepsilon$.

Proof: This result is an immediate consequence of Lemmas 5.2 and 5.3.

Lemma 5.5:

$$\sum_{k=1}^{n} m_{nk}(x)(x-x_{nk})^2 = O\left(\frac{1}{n^2}\right), \quad \text{as} \quad n \to \infty,$$

uniformly for $x \in I_\varepsilon$.

Proof:

$$\sum_{k=1}^{n} m_{nk}(x)(x-x_{nk})^2 = \sum_{k=1}^{n} \frac{U_n^2(x)}{(n+1)^2} (1-x_{nk}^2)^2 \ell_{nk}^2(x)$$

$$\leq \frac{M_\varepsilon^2}{(n+1)^2} \sum_{k=1}^{n} \ell_{nk}^2(x) = O\left(\frac{1}{n^2}\right), \quad \text{as} \quad n \to \infty,$$

by Lemma 5.1.

Proof of Theorem 5.2: The result stated in Theorem 5.2 is an immediate consequence of Lemmas 5.4 and 5.5 and Corollary 2.1.

Theorem 5.3: Let $x_{n1}, \cdots, x_{nn}$ $(-1 < x_{nn} < \cdots < x_{n1} < 1)$ be the zeros of $P_n(x)$ $(n = 1,2,\cdots)$. For each odd $n \geq 3$ define $L_n : C[-1,1] \to C[-1 + n^{-(1/3)}, 1 - n^{-(1/3)}]$ by:

$$L_n f(x) = \frac{\sum_{k=1}^{n} m_{nk}(x) f(x_{nk})}{\sum_{k=1}^{n} m_{nk}(x)},$$

where

$$m_{nk}(x) = \frac{1-2x_{nk}x+x_{nk}^2}{1-x_{nk}^2}\,\ell_{nk}^4(x),$$

$$\ell_{nk}(x) = \frac{P_n(x)}{(x-x_{nk})P_n'(x_{nk})} .$$

Let $||\cdot||_{n,\infty}$ denote the sup norm on $[-1+n^{-(1/3)}, 1-n^{-(1/3)}]$. Then there exists a constant $A > 0$ and an integer n_1 so that for every odd $n \geq n_1$ and every $f \in C[-1,1]$,

$$||L_n f - f||_{n,\infty} \leq A\,\omega_f\left(\frac{1}{n}\right) .$$

Remarks: Notice that the error estimate applies on an increasing sequence of intervals whose union is $(-1,1)$. However, that this cannot be improved so that the error estimate applies to $[-1,1]$ is demonstrated by the following:

It will now be shown that there exists a function $f \in C[-1,1]$ s.t. $L_n f(1) \not\to f(1)$.

If $k \leq \frac{n}{2}$, then (Szegö [10], p.122)

$$\frac{(k-\frac{1}{2})\pi}{n} \leq \arccos x_{nk} \leq \frac{k\pi}{n+1} ,$$

and hence there exist constants $C_1 > C_2 > 0$ s.t. for every n,k with $k \leq \frac{n}{2}$,

$$C_2\left(\frac{k}{n}\right)^2 \leq 1 - x_{nk} \leq C_1\left(\frac{k}{n}\right)^2 .$$

Also (Szegö [10, p.238), if $k \leq \frac{n}{2}$,

$$P_n'(x_k) \sim k^{-(3/2)}n^2 ,$$

and

$$1 \le 1 + x_{nk} \le 2 \ ,$$

so if $k \le n/2$, $m_{nk}(1) = \dfrac{1 - 2x_{nk} + x_{nk}^2}{1 - x_{nk}^2} \ell_{nk}^4(1) \sim \dfrac{1}{n^2}$.

Thus there exist constants $d_1 > d_2 > 0$ s.t. for every n,k with $k \le \frac{n}{2}$,

$$\frac{d_1}{n^2} > m_{nk}(1) > \frac{d_2}{n^2} \ .$$

Let $0 < \alpha < 1$ s.t. $\text{card}\,\{k | x_{nk} \ge \alpha\} \le \dfrac{d_2 n}{2d_1}$, for every $n = 1,2,\cdots$.

Define

$$f(x) = \begin{cases} 1 \ , & x = 1 \\ 0 \ , & x \in [-1,\alpha] \\ \dfrac{x-\alpha}{1-\alpha} \ , & x \in (\alpha,1) \end{cases}$$

Then $f \in C[-1,1]$ and

$$L_n f(1) \le \frac{\sum\limits_{x_k \ge \alpha} \frac{d_1}{n^2} f(x_{nk})}{\sum\limits_{k=1}^{n} \frac{d_2}{n^2}} \le \frac{1}{2} \ ,$$

and so

$$L_n f(1) \not\to f(1).$$

<u>Proof of Theorem 5.3</u>: Some of the simpler details will be omitted from the proof.

First, let $\xi_{n,n-1} < \cdots < \xi_{n1}$ be the zeros of $P_n'(x)$. It is easily proven that, for each $n > 3$ and $2 \le k \le n-1$,

$$\text{(I)} \qquad \min_{x \in [\xi_{nk}, \xi_{n,k-1}]} \ell_{nk}(x) = \min\{\ell_{nk}(\xi_{nk}), \ \ell_{nk}(\xi_{n,k-1})\} > 0.$$

Certain inequalities will be needed, and are listed below:

(1) (Balázs and Turán [2]). If n is odd, $n \geq 3$, then

$$|P_n(\xi_{nk})| \geq (8\pi j_{nk})^{-(1/2)}, \quad \text{where} \quad j_{nk} = \min(k,n+1-k).$$

(2) (Szegö [10], p.238).

$$|P_n'(x_{nk})| \sim j_{n+1,k}^{-(3/2)}\, n^2 .$$

(3) (Szegö [10], p.122).

$$\frac{(k-\frac{1}{2})\pi}{n} \leq \text{arc}\cos x_{nk} \leq \frac{k\pi}{n+1}, \quad k \leq [\tfrac{n}{2}],$$

and a symmetric result holds for the values $k > \frac{n}{2}$.

Easy calculations yield

$$|x_{nk} - x_{n,k+1}| \leq (4j_{nk}+1)\frac{\pi^2}{n^2} \quad (k = 1,\cdots,n-1)$$

and

$$|1 - x_{nk}^2| \geq \left(\frac{j_{n+1,k}}{n}\right)^2 \quad (k = 1,\cdots,n).$$

(4) (Szegö [10, p.165).

$$|P_n(x)| \leq \sqrt{\frac{2}{\pi}}\; n^{-(1/2)}(1-x^2)^{-(1/4)}.$$

Using these inequalities, it can be shown that there exists a constant $C > 0$ s.t., for every odd $n \geq 3$,

(II) $$\min\{\ell_{nk}(\xi_{nj}) \mid k = 2,\cdots,n-1;\ j = k-1,k\} \geq C.$$

For example, let $2 \leq k \leq n-1$. Then there exists a constant $C_0 > 0$ s.t.

$$\ell_{nk}(\xi_{nk}) = \frac{|P_n(\xi_{nk})|}{|\xi_{nk}-x_{nk}||P_n'(x_{nk})|} \geq C_o \frac{(8\pi j_{nk})^{-(1/2)} j_{n+1,k}^{3/2} n^2}{(4j_{nk}+1)\ \pi^2\ n^2}$$

$$= \frac{C_o}{2\sqrt{2}\ \pi^{5/2}} \left(\frac{j_{n+1,k}^{3/2}}{(4j_{nk}+1)\sqrt{j_{nk}}}\right) \geq \frac{C_o}{40\pi^{5/2}} > 0.$$

The treatment of $\ell_{nk}(\xi_{n,k-1})$ is similar.

From (I) and (II) it follows by easy calculations that there exists $C_1 > 0$ s.t. for odd $n \geq 3$,

(III) $$\min_{x\in[\xi_{n,n-1},\xi_{n1}]} \sum_{k=1}^{n} m_{nk}(x) \geq C_1 .$$

Finally, we will show that

(IV) $$\sup_{|x|\leq 1-n^{-(1/3)}} \sum_{k=1}^{n} m_{nk}(x)(x-x_{nk})^2 = 0\left(\frac{1}{n^2}\right), \quad \text{odd } n \to \infty.$$

Let n be odd, $n \geq 3$, and $x \in [-1+n^{-(1/3)}, 1-n^{-(1/3)}]$. The sum will be considered in two parts - those k for which $|x-x_{nk}| \leq \frac{n^{-(1/3)}}{2}$, and those k for which $|x-x_{nk}| > \frac{n^{-(1/3)}}{2}$. Let $\alpha_n = \frac{1}{2} n^{-(1/3)}$. If

$$h_{nk}(x) = \left(\frac{1 - 2x_{nk}\ x + x_{nk}^2}{1-x_{nk}^2}\right) \ell_{nk}^2(x) ,$$

then

$$\sum_{k=1}^{n} h_{nk}(x) \equiv 1.$$

$$\sum_{|x-x_{nk}|\leq\alpha_n} m_{nk}(x)(x-x_{nk})^2 = \sum_{|x-x_{nk}|\leq\alpha_n} \frac{h_{nk}(x)P_n^2(x)}{(P_n'(x_{nk}))^2}$$

$$= 0\left(\frac{1}{n^2}\right) \sum_{|x-x_{nk}|\leq\alpha_n} h_{nk}(x) \left(\frac{j_{n+1,k}}{n}\right)^3 (1-x^2)^{-(1/2)}$$

$$= O\left(\frac{1}{n^2}\right) \sum_{|x-x_{nk}|\leq\alpha_n} h_{nk}(x) \left(\frac{j_{n+1,k}}{n}\right)^3 (1-x_{nk}^2)^{-(1/2)}$$

(since $(1-x^2)^{-(1/2)} = (1-x_{nk}^2)^{-(1/2)} \left(\frac{1-x^2}{1-x_{nk}^2}\right)^{-(1/2)} = O\left((1-x_{nk}^2)^{-(1/2)}\right)$.)

$$= O\left(\frac{1}{n^2}\right) \sum_{k=1}^{n} h_{nk}(x)$$

$$= O\left(\frac{1}{n^2}\right) .$$

Also, quite easily one gets

$$\sum_{|x-x_{nk}|>\alpha_n} m_{nk}(x)(x-x_{nk})^2 = \sum_{|x-x_{nk}|>\alpha_n} \frac{(1-2x_{nk}x+x_{nk}^2)P_n^4(x)}{(1-x_{nk}^2)(x-x_{nk})^2\left(P_n'(x_{nk})\right)^4}$$

$$= O\left(\frac{1}{n^2}\right) ,$$

establishing IV.

Notice that certainly $[-1+n^{-(1/3)}, 1-n^{-(1/3)}] \subset [\xi_{n,n-1}, \xi_{n1}]$, so Theorem 5.3 now follows immediately from III, IV, and Corollary 2.1.

CHAPTER 6

CONCLUSION

As predicted in Chapter 1, the error estimates achieved in Chapter 5 are an improvement over those obtained for H-F interpolation. However, it is not known whether the results of Chapter 5 can be improved. It is a simple matter to show that if $\{L_n\}$ is a sequence of proper interpolation operators defined on $C[-1,1]$ and $\{\varepsilon_n\}$ is a sequence s.t. $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$, with

$$\overline{\lim_{n\to\infty}} \sup_{f\in C[-1,1]} \left\{ \frac{\|L_n f - f\|_\infty}{\omega_f(\varepsilon_n)} \right\} = O(1) ,$$

then $\varepsilon_n \geq K \cdot \frac{1}{n}$ for some constant K and all n. ($\|\cdot\|_\infty$ denotes the sup norm on any nontrivial subinterval I of $[-1,1]$.). However, it is not known whether $\varepsilon_n \sim \frac{1}{n}$ is the best possible in order that

$$\sup_{f\in C[-1,1]} \overline{\lim_{n\to\infty}} \left\{ \frac{\|L_n f - f\|_\infty}{\omega_f(\varepsilon_n)} \right\} = O(1) ,$$

even if $\{L_n\}$ is restricted to some smaller class of interpolation operators which contains those of Chapter 5.

It seems likely that the proof of Lemma 4.1 could be strengthened so that the $\sqrt{\ln \ln n}$ factor could be eliminated. It also seems likely that all the results of Chapter 5 could be combined into a general theorem using ultraspherical polynomials as the building blocks for some rational interpolation operator.

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